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On a Problem of Rivlin in L_1 Approximation*

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This paper gives the answer to a problem of Rivlin in L_1 approximation in the case when $n = 2$.

I. INTRODUCTION

In a conference held in Oberwolfach in 1968, Rivlin [1] proposed the following problem:

Characterize those n-tuples of algebraic polynomials $\{p_0, p_1, ..., p_{n-1}\}$ with degrees satisfying

$$\deg p_j = j \qquad (j = 0, 1, ..., n-1),$$

for which there exists an $f \in C[-1, 1]$ such that the polynomial of best uniform approximation of degree j to f is p_j $(j = 0, 1, ..., n-1)$. What is the characterization in the particular case when $n = 2$?

Several authors [2–9] have studied this problem. In this paper we consider the above problem in $C[-1, 1]$ with the L_1 norm:

$$\|f\| = \int_{-1}^{1} |f(x)|\, dx,$$

and give the answer in the particular case when $n = 2$. That is the following:

THEOREM. *Let P_j be the set of polynomials of degree $\leqslant j$ $(j = 0, 1)$, and $p_j \in P_j$ $(j = 0, 1)$. Then there exists an $f \in C[-1, 1]$ such that p_j is a best approximation to f from $P_j (j = 0, 1)$ if and only if the polynomial $p = p_1 - p_0$ changes sign once in $[-1, 1]$ or is identically equal to zero.*

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Before proving the theorem we introduce some notation.

$$Z_+(h) = \{x \in [-1, 1]: h(x) > 0\},$$
$$Z_-(h) = \{x \in [-1, 1]: h(x) < 0\},$$
$$Z(h) = \{x \in [-1, 1]: h(x) = 0\},$$
$$m(E) = \text{the Lebesgue measure of the set } E.$$

II. Proof of the Theorem

We can suppose without loss of generality that $p_0 = 0$.

Necessity. Assume that there exists an $f \in C[-1, 1]$ such that p_j is a best approximation to f from P_j $(j = 0, 1)$, where $p_0 = 0$ and $p_1 = p$, but the condition of the theorem is not satisfied, i.e., $p \neq 0$ and does not change sign in $[-1, 1]$, say $p \geqslant 0$ on $[-1, 1]$.

By Theorem 4.2 in [10] we have

$$\left| \int_{-1}^{1} \operatorname{sgn} f(x)\, dx \right| \leqslant \int_{Z(f)} dx,$$
$$\left| \int_{-1}^{1} \operatorname{sgn}(f(x) - p(x))\, dx \right| \leqslant \int_{Z(f-p)} dx,$$

i.e.,

$$|m(Z_+(f)) - m(Z_-(f))| \leqslant m(Z(f)),$$
$$|m(Z_+(f-p)) - m(Z_-(f-p))| \leqslant m(Z(f-p)).$$

Hence,

$$m(Z_+(f)) - m(Z_-(f)) - m(Z(f)) \leqslant 0, \tag{1}$$
$$m(Z_+(f-p)) - m(Z_-(f-p)) + m(Z(f-p)) \geqslant 0. \tag{2}$$

On the other hand since

$$Z_-(f) \cup Z(f) \subset Z_-(f-p) + e$$

and

$$Z_+(f-p) \cup Z(f-p) \subset Z_+(f) + e, \tag{3}$$

where $e = \{-1, 1\}$,

$$m(Z_-(f)) + m(Z(f)) \leqslant m(Z_-(f-p)),$$

and

$$m(Z_+(f-p)) + m(Z(f-p)) \leqslant m(Z_+(f)),$$

whence

$$m(Z_+(f-p)) - m(Z_-(f-p)) + m(Z(f-p)) \\ \leqslant m(Z_+(f)) - m(Z_-(f)) - m(Z(f)). \tag{4}$$

But from (1), (2) and (4) it follows that

$$m(Z_+(f-p)) - m(Z_-(f-p)) + m(Z(f-p)) \\ = (Z_+(f)) - m(Z_-(f)) - m(Z(f)) = 0,$$

so

$$m(Z(f)) + m(Z_-(f)) = m(Z_+(f)) = 1,$$
$$m(Z(f-p)) + m(Z_+(f-p)) = m(Z_-(f-p)) = 1.$$

Thus,

$$m(Z(f-p)) + m(Z_+(f-p)) = m(Z_+(f)). \tag{5}$$

From (3) and (5) we obtain

$$m(Z_+(f)\backslash(Z(f-p) \cup Z_+(f-p))) = 0,$$

a contradiction, because the set

$$Z_+(f)\backslash(Z(f-p) \cup Z_+(f-p)) = Z_+(f) \cap Z_-(f-p) \\ = \{x \in [-1, 1]: 0 < f(x) < p(x)\}$$

is nonempty. In fact, if $f \leqslant 0$ (or $f \geqslant p$), p (or 0) could not be a best approximation to f from P_1 (or P_0).

Sufficiency. Assume now that $p = ax + b$ changes sign once in $[-1, 1]$, because the theorem is obviously valid for $p = 0$. Then $a \neq 0$ and $t = -b/a \in (-1, 1)$. We discuss three cases.

Case 1. $t < 0$. Set

$$u = \min\{t, -\tfrac{1}{2}\}, \qquad v = \max\{t, -\tfrac{1}{2}\}, \qquad w = \tfrac{1}{4}\min\{t+1, -t\};$$

$$x_0 = -1, \qquad x_1 = u - w, \qquad x_2 = -\tfrac{1}{2}, \qquad x_3 = v + w,$$
$$x_4 = \tfrac{1}{2} - w, \qquad x_5 = \tfrac{1}{2}, \qquad x_6 = 1,$$

and

$$\begin{aligned} f(x) &= 0, & x &= x_0, x_1 \\ &= p(x_2), & x &= x_2 \\ &= 0, & x &= x_3, x_4 \\ &= p(x_5), & x &= x_5 \\ &= 2p(x_6), & x &= x_6 \end{aligned}$$

linear for the other x.

Firstly, since $m(Z(f)) = (x_1 - x_0) + (x_4 - x_3) = 3/2 + u - v - 3w > 1$, by Theorem 4.2 in [10], 0 is a best approximation to f from P_0.

Secondly, it is easy to see by simple calculation that

$$\begin{aligned} \operatorname{sgn}(f(x) - p(x)) &= \operatorname{sgn} a, & x &= x_0, x_1 \\ &= 0, & x &= x_2 \\ &= -\operatorname{sgn} a, & x &= x_3, x_4 \\ &= 0, & x &= x_5 \\ &= \operatorname{sgn} a, & x &= x_6. \end{aligned}$$

We have

$$\begin{aligned} \operatorname{sgn}(f(x) - p(x)) &= \operatorname{sgn} a, & |x| &> \tfrac{1}{2} \\ &= -\operatorname{sgn} a, & |x| &< \tfrac{1}{2}. \end{aligned}$$

Thus, for any $q \in P_1$

$$\int_{-1}^{1} q(x) \operatorname{sgn}(f(x) - p(x))\, dx$$

$$= (\operatorname{sgn} a) \left(\int_{-1}^{-1/2} q(x)\, dx - \int_{-1/2}^{1/2} q(x)\, dx + \int_{1/2}^{1} q(x)\, dx \right) = 0,$$

and p is a best approximation to f from P_1.

Case 2. $t = 0$.

$f = p$ has best approximations 0 and p from P_0 and P_1, respectively.

Case 3. $t > 0$.

Consider $p^*(x) = p(-x)$. Then $t^* = -t < 0$. According to Case 1 above, there exists an f^* with 0 and p^* as its best approximations from P_0 and P_1, respectively. Hence $f(x) \equiv f^*(-x)$ has best approximations 0 and $p(x) \equiv p^*(-x)$.

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